can be performed with respect to its elements. We then have to satisfy the condition that B must be positive definite. A more sensible method is to use our above method of constructing the extremal sets of stability and write a problem of type (4.4) /3/.

To solve problem (4.4) we used iterative algorithms of type (3.5), taking the case when the maxima with respect to t, x_0, α are not unique. When solving this class of problems we found a high rate of convergence of the iterative processes to the point $\alpha^{(0)}$. The modes obtained were mainly analysed for optimality from the physical stand-point. It must be said that the optimal design of acceleration and focussing systems by our method enables the efficiency of such devices to be greatly improved /3/.

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ON A REMARK OF POINCARE*

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Descriptions of non-autonomous mechanical systems by Poincaré's equations /1/ in the Lagrangian and canonical forms are studied. For systems with a Hamiltonian which depends only on the Chetayev variables /2/ and time, the existence of a complete set of linear (non-commuting) first integrals is proved. The required conditions imposed on the kinetic energy and active forces are studied. Explicit relations for evaluating the integrals by quadratures are obtained. The connection of Poincaré's equations with system of hydrodynamic type is noted. The case of the motion of autonomous mechanical systems when the Lagrange function, expressed in velocity parameters, is independent of the coordinates, was mentioned by Poincaré as being of special interest. This case includes the theory of geodesic left-invariant metrics in Lie groups (a "generalized rigid body" /3/). The primary element of its construction is a Lie group (configuration manifold). Every metric which is defined in it and is invariant under the group operations, defines the kinetic energy. In studies not directly connected with Poincaré's remark, the initial object is the mechanical

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system (the kinetic energy and active forces). Therein lies the difference between the statements of the problems and the results obtained.

1. Initial relations.

Let us summarize the facts used below, which we stated and proved in our paper "Some properties and applications of the Poincare-Chetayev equations*. (*Preprint 273, Inst. Problem Mechaniki, Akad. Nauk SSSR, Moscow, 1968). Throughout, the indices take the values

$$i, j, l, k, p = 1, \ldots, n; \quad \alpha, \beta = 1, \ldots, N; \quad \nu = 1, \ldots, N - s$$

Summation over repeated indices is understood.

Let x_α be the coordinates of the mechanical system with s degrees of freedom, constrained by $N - s$ ideal holonomic non-stationary couplings $f_\nu(x, t) = 0$, the explicit form of which is not necessarily known. We assume that the possible displacements and actual velocities of the system can be written as

$$\delta x_\alpha = \xi_\alpha^{\;j}(x, t)\,\Omega_j, \quad x_\alpha^{\cdot} = \xi_\alpha^{\;j}(x, t)\,\eta_j + \xi_\alpha(x, t); \quad N, n \geqslant s = \operatorname{rank} \| \xi_\alpha^{\;j} \|$$

where the parameters Ω_j of the possible displacements and the Poincaré parameters η_j are independent. It is assumed that the corresponding operators

$$X_0 = \partial/\partial t + \xi_\alpha(x, t)\,\partial/\partial x_\alpha, \quad X_j = \xi_\alpha^{\;j}(x, t)\,\partial/\partial x_\alpha$$

of which just $s + 1$ are linearly independent, form a basis of the $(n + 1)$-dimensional Lie algebra (algebra A)

$$[X_k, X_l] = c_{kl}^p X_p, \quad [X_k, X_0] = c_{k0}^p X_p$$

Let $L(t, x, x^{\cdot})$ be the Lagrange function of the system and $L^*(t, x, \eta) \equiv L(t, x, \xi_1^{\;j}\eta_j + \xi_1, \ldots, \xi_N^{\;j}\eta_j + \xi_N)$; Q_α be the non-conservative forces acting on the system. The equations of motion

$$\frac{d}{dt}\left(\frac{\partial L^*}{\partial \eta_j}\right) = X_j' L^* + \xi_\alpha^{\;j} Q_\alpha, \quad X_j' = X_j + (c_{0j}^p + c_{lj}^p \eta_l)\frac{\partial}{\partial \eta_p} \tag{1.1}$$

$$x_\alpha^{\cdot} = \xi_\alpha^{\;j}(x, t)\,\eta_j + \xi_\alpha(x, t) \tag{1.2}$$

contain $N - s$ redundant coordinates x and $n - s$ redundant parameters η. With $N = n = s$, $c_{0j}^{\;p} = 0$, $Q_\alpha = 0$, $\partial \xi_\alpha^{\;j}/\partial t = 0$, $\xi_\alpha = 0$ they become Poincaré's equations. With $N \geqslant s$, $n = s$, $c_{0j}^{\;p} = 0$, $Q_\alpha = 0$ they become the Poincaré-Chetayev equations for the case of non-stationary couplings /2/.

We note some properties of Eqs.(1.1).

1°. Of the n equations (1.1), only s are independent. The rest are linear combinations of the independent equations. Hence, if the latter correspond to the parameters $\eta_1, \ldots, \eta_s$, then the parameters $\eta_{s+1}, \ldots, \eta_n$ are free, i.e., they are not connected by any auxiliary conditions.

2°. The operators X_j' form a basis of an algebra isomorphic to algebra A: $[X_j', X_k'] = c_{jk}^{\;p} X_p'$.

3°. We can obtain Eqs.(1.1) from the known Lagrange equations by a passage to the quasi-velocities in accordance with (1.2).

We pass in Eqs.(1.1) and (1.2) with $n = s$ to Chetayev variables $y_i = \partial L^*/\partial \eta_i$. In mechanical problems the Lagrange function L^* is non-degenerate quadratic form of the variables $\eta_1, \ldots, \eta_s$. By the property of the Legendre transformation (Donkin's theorem /4/) there is an inverse transformation $\eta_i = \partial H^*/\partial y_i$, generated by the function $H^* = \eta_i y_i - L^*$. The equations of motion take the canonical form

$$\begin{aligned} x_\alpha^{\cdot} &= \xi_\alpha^{\;i}\frac{\partial H^*}{\partial y_i} + \xi_\alpha \\ y_j^{\cdot} &= -\left(\xi_\alpha^{\;j}\frac{\partial H^*}{\partial x_\alpha} + c_{jl}^{\;p} y_p \frac{\partial H^*}{\partial y_l}\right) + c_{0j}^{\;p} y_p + \xi_\alpha^{\;j} Q_\alpha \end{aligned} \tag{1.3}$$

Chetayev (/2/, p.199) was the first to obtain Eqs.(1.3) in the redundant coordinates with non-stationary couplings under the assumption that there are no non-conservative forces ($Q_\alpha = 0$) and that the operator X_0 permutes with the operators $X_1, \ldots, X_s$ ($c_{0j}^{\;p} = 0$). If $n = s$ and $X_0 = 0$, Eqs.(1.1) can be obtained directly from Hamilton's equations

$$x_i^{\cdot} = \partial H/\partial p_i, \quad p_i^{\cdot} = -\partial H/\partial x_i + Q_i$$

by replacing the momenta $y_k = \xi_i^{\;k} p_i$.

We can write Eqs.(1.3) in the form

$$x_\alpha^{\cdot} = Y_\alpha^* H^* + \xi_\alpha, \quad y_j^{\cdot} = -X_j^* H^* + c_{0j}^p y_p + \xi_\alpha^{\ j} Q_\alpha \tag{1.4}$$
$$Y_\alpha^* = \xi_\alpha^{\ i}(x, t)\frac{\partial}{\partial y_i}, \quad X_j^* = \xi_\alpha^{\ j}(x, t)\frac{\partial}{\partial x_\alpha} + c_{jl}^p y_p \frac{\partial}{\partial y_l}$$

By the operator of shift along the trajectories of motion of the system $z_i^{\cdot} = f_i(z_i, t)$ we henceforth mean the operator of total time differentiation in the light of the equations of motion of functions which are given in the space $\{z, t\}$, $S = \partial/\partial t + f_i(z, t)\,\partial/\partial z_i$, or the operator acting on functions which are defined in the phase space $\{z\}$, $S = f_i(z)\,\partial/\partial z_i$.

We consider a mechanical system with ideal holonomic non-stationary couplings, whose motion is described by Eqs.(1.4). The shift operator of this system is

$$S = \frac{\partial}{\partial t} + (Y_\alpha^* H^* + \xi_\alpha)\frac{\partial}{\partial x_\alpha} + (-X_j^* H^* + c_{0j}^p y_p + \xi_\alpha^{\ j} Q_\alpha)\frac{\partial}{\partial y_j}$$

We will list some of its properties.

1°. The operator S can be written as

$$S = X_0^* - \frac{\partial H^*}{\partial x_\alpha} Y_\alpha^* + \frac{\partial H^*}{\partial y_j} X_j^* \tag{1.5}$$
$$X_0^* = X_0 + \left(c_{0j}^p y_p + \xi_\alpha^{\ j} Q_\alpha\right)\frac{\partial}{\partial y_j}$$

Thus S belongs to the linear hull stretched on operators X_0^*, Y_α^*, X_j^*.

2°. The system of operators X_0^*, Y_α^*, X_j^* is closed. When there are no non-conservative forces $(Q_\alpha = 0)$ its multiplication table is given by the commutation relations

$$[X_j^*, X_k^*] = c_{jk}^p X_p^*, \quad [Y_\alpha^*, Y_\beta^*] = 0 \tag{1.6}$$
$$[X_j^*, X_\alpha^*] = \frac{\partial \xi_\alpha^{\ j}}{\partial x_\beta} Y_\beta^*, \ [X_0^*, \quad \alpha^*] = \frac{\partial \xi_\alpha}{\partial x_\beta} Y_\beta^*$$
$$[X_0^*, X_k^*] = c_{0k}^p X_p^*$$

3°. If the forces $Q_\alpha \neq 0$ and are independent of the variables y, the last set of relations in Para.5 will be replaced by

$$[X_0^*, X_k^*] = c_{0k}^p X_p^* - \left(X_k Q_\beta' + \frac{\partial \xi_\alpha^{\ k}}{\partial x_\beta} Q_\alpha'\right) Y_\beta^* \tag{1.7}$$
$$Q_\beta' = Q_\beta - \partial(b_0 + U)/\partial x_\beta$$

where $(b_0 + U)$ are the terms of H^*, independent of y.

4°. The left-hand sides of the coupling equations belong to the kernel K of the operators X_0^*, Y_α^*, X_j^*, i.e.,

$$X_0^* f_\nu = Y_\alpha^* f_\nu = X_j^* f_\nu = 0$$

5°. Let $x_{\sigma+1}, \ldots, x_s$ be the coordinates of the conservative system such that we have

$$\partial H^*/\partial x_{\sigma+1} = \ldots = \partial H^*/\partial x_s = 0, \quad \partial\xi_\gamma/\partial x_{\sigma+1} = \ldots$$
$$\ldots = \partial\xi_\gamma/\partial x_s = 0$$
$$\partial\xi_\gamma^{\ j}/\partial x_{\sigma+1} = \ldots = \partial\xi_\gamma^{\ j}/\partial x_s = 0; \quad \gamma = 1, \ldots, \sigma$$

The rank of the matrix $\| \xi_j^{\ i} \|$, from which the block

$$\begin{Vmatrix} \xi_{\sigma+1}^1 & \ldots & \xi_{\sigma+1}^s \\ \cdot & \ldots & \cdot \\ \xi_s^{\ 1} & \ldots & \xi_s^{\ s} \end{Vmatrix}$$

is removed, is equal to σ, and

$$\operatorname{rank}\begin{Vmatrix} \xi_1^{\ 1} & \ldots & \xi_1^{\ s} \\ \cdot & \ldots & \cdot \\ \xi_\sigma^{\ 1} & \ldots & \xi_\sigma^{\ s} \end{Vmatrix} = \sigma$$

Then, in accordance with (1.5) and (1.6), the kernel K consists of functions f_γ and of a further $s - \sigma$ functions ω_μ

$$Y_\gamma^*\omega_\mu = X_j^*\omega_\mu = X_0^*\omega_\mu = 0; \quad \gamma = 1, \ldots, \sigma; \quad j = 1, \ldots, s; \quad \mu = 1, \ldots, s - \sigma \tag{1.8}$$

These are the functionally independent first integrals of the mechanical system. If, in the Lagrangian form, the latter has the cyclic coordinates $x_{\sigma+1}, \ldots, x_{\sigma+m}$ $(m \leqslant s - \sigma)$, which, on passage to the canonical form, are included in the Chetayev coordinates $x_{\sigma+1}, \ldots, x_s$, then all the corresponding cyclic integrals are contained among the functions ω_μ.

Notes. 1°. Here and henceforth, by a first integral of motion ω_μ, we shall mean a solution of the equation $S\omega = 0$. The uniqueness of this solution and the degree to which this definition corresponds to modern geometrical ideas on first integrals will not be discussed.

2°. The condition on the rank of the matrix $\|\xi_j^i\|$ is satisfied automatically and in an obvious way in two cases: with $N = s$ and when $x_1, \ldots, x_\sigma$ are the Lagrangian coordinates, and $x_{\sigma+1}, \ldots, x_N$ are parametrized by the remaining $s - \sigma$ Lagrangian coordinates of the system.

A special case, noted by Poincaré, corresponds to the following situation which is further considered in Paras.2-4. The mechanical system with ideal holonomic non-stationary couplings and with s degrees of freedom, moves in the absence of active forces $(U = 0,\ Q_\alpha = 0)$. We will assume that, by introducing a suitable algebra, we can remove all the coordinates from the function H^*: $H^* = H^*(t, y)$. Then the shift operator S is an element of the linear hull, stretched over the basis system of operators $X_0^*, X_1^*, \ldots, X_s^*$ of the $(s+1)$-dimensional Lie algebra

$$S = X_0^* + \frac{\partial H^*}{\partial y_j} X_j^*, \quad [X_j^*, X_k^*] = c_{jk}^p X_p^*, \quad [X_0^*, X_k^*] = c_{0k}^p X_p^*$$

This follows from (1.5) and (1.6). Putting

$$H^* = \tfrac{1}{2} b_{ij}(t)\, y_i y_j + b_i(t)\, y_i + b_0(t) \tag{1.9}$$

in Eqs.(1.4) we obtain with $N = s$

$$\begin{aligned} x_\alpha^\cdot &= \xi_\alpha^i b_{ij} y_j + \xi_\alpha^i b_i + \xi_\alpha \\ y_j^\cdot &= c_{lj}^p b_{li} y_p y_i + (c_{0j}^p + c_{lj}^p b_l)\, y_p \end{aligned} \tag{1.10}$$

2. On the existence and evaluation of the first integrals. By (1.8), the system of equations

$$X_0^*\omega = X_1^*\omega = \ldots = X_s^*\omega = 0 \tag{2.1}$$

is compatible and its solutions are the first integrals of the equations of motion (1.10). The closed system of operators $X_0^*, X_1^*, \ldots, X_s^*$ acts in $(2s+1)$-dimensional space $\{t, x, y\}$, and hence has precisely $2s + 1 - (s + 1) = s$ independent solutions. Let us show that they are all linear in the variables y_j. The integrals that depend only on y_j are non-linear. For $s = 3$ they can be evaluated in explicit form (/5/, p.51). We shall seek the compatible solutions of system (2.1) in the form $\omega = \mu_k y_k$, $\mu_k = \mu_k(t, x)$. For the functions μ_k we obtain the system of equations

$$X_0\mu_k = c_{p0}^k\ \mu_p, \quad X_\nu\mu_k = c_{p\nu}^k\ \mu_p \tag{2.2}$$

To will show that it is compatible, we write the compatibility conditions for each pair of equations $X_0\mu_k - c_{p0}^k\ \mu_p = 0$, $X_\alpha\mu_k - c_{p\alpha}^k\ \mu_p = 0$, and obtain by Jacobi's conditions

$$[X_\alpha, X_\beta]\,\mu_k - c_{p\beta}^k X_\alpha \mu_p + c_{p\alpha}^k X_\beta \mu_p = (c_{\alpha\beta}^p c_{lp}^k + c_{l\alpha}^p c_{\beta p}^k + c_{\beta l}^p c_{\alpha p}^k)\,\mu_l = 0,$$

and similar relations with the replacement $\alpha \to 0$, $\beta \to \nu$.

By successive integration of system (2.2) we can show that it has s linearly unconnected solutions $(\mu_1^{(k)}, \ldots, \mu_s^{(k)})$, $k = 1, \ldots, s$, which form a fundamental system $(\mu_j^k = \delta_j^k$ at a point (t_0, x^0) of general position), while the general solution of system (2.2) is a linear combination of them with constant coefficients.

3. Evaluation of linear integrals. We will show that, if the algebra A with basis

$$X_1, \ldots, X_s \tag{3.1}$$

is solvable (/6/, p.208), the linear integrals of motion can be evaluated by quadratures. We are thus concerned with solvability in the quadratic Eqs.(2.2).

Let $(\mu_1^i, \ldots, \mu_s^i)$ be s linearly independent solutions of system (2.2) (which do not necessarily form a fundamental system). We change to a new system of functions μ_k^{*p}, given by the relations $\mu_k^{*\beta}\mu_\beta^l = \delta_k^l$, $\det\|\mu_\beta^l\| \neq 0$. These functions satisfy the equations

$$X_0\mu_k^{*\beta} = c_{0\beta}^p\mu_k^{*p}, \quad X_j\mu_k^{*\beta} = c_{j\beta}^p\mu_k^{*p}, \quad \beta, k, p, j = 1, \ldots, s \tag{3.2}$$

which are obtained from (2.2) by multiplication by the function $\mu_i^{*\beta}\,\mu_l^{*k}$ and addition. Systems (3.2) and (2.2) are equivalent.

If the algebra (3.1) is solvable, then the $(s+1)$-dimensional algebra obtained from it by the addition of the vector X_0 is also solvable (it is clear from the commutation relations $[X_0, X_k] = c_{0k}^p X_p$ that algebra (3.1) forms an ideal (/6/, p.143). In a solvable algebra we can find a new basis

$$X_1^\circ, \ldots, X_s^\circ, X_{s+1}^\circ = X_0 \tag{3.3}$$

such that every system of operators $X_1^\circ, \ldots, X_i^\circ$ $(i = 1, \ldots, s+1)$ corresponds to a normal divisor of the local group with algebra (3.3) (/6/, p.212). Consequently, the commutation relations will have the form

$$[X_k^\circ, X_j^\circ] = c_{kj}^1 X_1^\circ + c_{kj}^2 X_2^\circ + \ldots + c_{kj}^j X_j^\circ, \quad k, j = 1, \ldots, s+1$$

i.e., $c_{kj}^p = 0$ for $k < p$ or $j < p$. In the basis (3.3), Eqs.(3.2) become

$$\begin{aligned} &X_j^\circ\mu_k^{*1} = c_{j1}{}^1\mu_k^{*1}, \quad X_j^\circ\mu_k^{*2} = c_{j2}{}^1\mu_k^{*1} + c_{j2}{}^2\mu_k^{*2}, \ldots \\ &X_j^\circ\mu_k^{*s} = c_{js}{}^1\mu_k^{*1} + c_{js}{}^2\mu_k^{*2} + \ldots + c_{js}{}^s\mu_k^{*s}, \quad k = 1, \ldots, s; \\ &j = 1, \ldots, s+1 \end{aligned} \tag{3.4}$$

The system written can be integrated successively, starting with the first

$$\mu_k^{*1} = c_k{}^1 \exp c_{j1}{}^1 \int \alpha_j{}^\beta\, dx_\beta, \; x_{s+1} \equiv t, \; \xi_j^{s+1} \equiv \xi_j, \quad c_k{}^1 = \mathrm{const}\,(\alpha_j{}^\beta \xi_\beta{}^k \delta_j{}^\kappa)$$

After integration of the first i subsystem of (3.4) the $(i+1)$-th subsystem consists of compatible equations of the type

$$\partial z/\partial x_j = a_j z + b_j \tag{3.5}$$

where a_j, b_j are known functions of $x_1, \ldots, x_{s+1}$. This system is integrated by variation of the arbitrary constant

$$\begin{aligned} &z = C^* z_0, \quad z_0 = \exp\left(\int a_j dx_j\right) \\ &C^* = C + \int b_j z_0^{-1} dx_j, \quad C = \mathrm{const} \end{aligned}$$

since it follows from the compatibility conditions of Eqs.(3.5) that $a_j dx_j$ and $b_j z_0^{-1} dx_j$ are total differentials.

The set of functions $\mu_k{}^{*i}$ thus obtained will depend on s^2 arbitrary constants c_j^i, which will be chosen differently depending on the choice of the integral basis. We only require that this basis no non-degenerate. For this it suffices e.g., that the constants c_j^i be subject to the condition that, at a fixed point at which no quadrature has a singularity, we have $\mu_k{}^{*i} = \delta_k{}^i$. In some problems the conditions $c_j{}^i = \delta_j{}^i$ ensure the non-degeneracy of the intregal basis $(\det \|\mu_j{}^{*i}\| \neq 0)$ and simultaneously simplify the working. Sometimes it is simpler not to use the above procedure at all (it always gives an answer), but to evaluate the $\mu_k{}^i$ directly from (2.2). In short, we have proved the following theorem.

Theorem. A mechanical system with s degrees of freedom and holonomic ideal non-stationary couplings, whose kinetic energy in Chetayev variables is independent of the coordinates, has just s linear first integrals when it moves in the absence of active forces. If the Lie algebra used for passing to the Chetayev variables is solvable, these integrals can be found explicitly by quadratures.

Note that, if the algebra is commutative, we can find explicitly for autonomous systems a coordinate transformation which converts all s linear into cyclic integrals.

4. On the conditions for reducing the kinetic energy of a scleronomous system to a form independent of the coordinates.

Let the Lagrangian and function H^* be

$$\begin{aligned} &L = {}^1/_2 a_{ij}(t, x)\, x_i^{\cdot} x_j^{\cdot} + a_i(t, x)\, x_i^{\cdot} + a_0(t, x) - U(x) \\ &H^* = {}^1/_2 b_{ij}(t)\, y_i y_j + b_i(t)\, y_i + b_0(t) + U(x) \end{aligned} \tag{4.1}$$

Then,

$$b_{ij} = a^{kl}\alpha_i^{\ k}\alpha_j^{\ l}, \quad b_i = -(a^{kl}a_k + \xi_l)\alpha_i^{\ l} \tag{4.2}$$
$$b_0 = {}^1/_2 a^{kl}a_k a_l - a_0$$
$$a^{kl}a_{li} = \delta_i^{\ k}, \quad \alpha_i^{\ l}\xi_l^{\ k} = \delta_i^{\ k}$$

For the proof, we calculate

$$y_\sigma\eta_\sigma - L^* = {}^1/_2 a_{ij}(\xi_j^{\ \tau}\eta_\tau + \xi_j)\xi_i^{\ \sigma}\eta_\sigma - {}^1/_2 a_{ij}(\xi_j^{\ \tau}\eta_\tau + \xi_j)\xi_i - a_i\xi_i - a_0 + U$$

On replacing the terms on the right-hand side of this expression that contain η_σ in accordance with the relations

$$a_{kj}(\xi_j^{\ \tau}\eta_\tau + \xi_j) = y_\sigma\alpha_\sigma^{\ k} - a_k, \quad \xi_k^{\ l}\eta_l = a^{kl}(y_\sigma\alpha_\sigma^{\ l} - a_l) - \xi_k$$

we obtain relations (4.2), and also

$$H^* = {}^1/_2 a^{kl}(y_\sigma\alpha_\sigma^{\ k} - a_k)(y_\tau\alpha_\tau^{\ l} - a_l) - \alpha_\sigma^{\ k}\xi_k y_\sigma - a_0 + U$$

Clearly, H^* cannot be obtained from the Hamilton function of the system

$$H = {}^1/_2 a^{kl}(p_k - a_k)(p_l - a_l) - a_0 + U$$

by the replacement $p_k = y_i\alpha_i^{\ k}$, if $\Sigma\xi_i^2 \neq 0$; in this sense H^* is not the Hamiltonian of the system when $\Sigma\xi_i^2 \neq 0$.

We recall for convenience Maurer's relation (/6, p.104/)

$$\partial\alpha_\gamma^{\ \tau}/\partial x_j - \partial\alpha_\gamma^{\ j}/\partial x_\tau = c_{\mu\nu}^\gamma\alpha_\mu^{\ \tau}\alpha_\nu^{\ j} \tag{4.3}$$

Let the couplings imposed on the system be time-independent. Then, $a_i = a_0 = 0$, $\partial a_{ij}/\partial t = 0$. It can be assumed without loss of generality that $b_{ij} = \delta_j^{\ i}$ and $b_i = 0$ (taking $\xi_l = 0$). The first group of relations (4.2) gives

$$a_{i\tau} = \alpha_\gamma^{\ i}\alpha_\gamma^{\ \tau} \tag{4.4}$$

On writing the result of differentiating relations (4.4) in the light of (4.3), and of the relations $\xi_j^{\ k} = a^{\sigma j}\alpha_k^{\ \sigma}$, after introducing the Christoffel symbol of the second kind and transforming the terms which contain the structural constants, we find by replacement of the indices that

$$\partial\alpha_k^{\ i}/\partial x_\tau = \Gamma_{i\tau}^\beta\alpha_k^{\ \beta} + \overset{*}{c}{}_{\mu k}^\gamma\alpha_\gamma^{\ i}\alpha_\mu^{\ \tau}, \quad \overset{*}{c}_{\mu k} = {}^1/_2(c_{\mu k}^\gamma + c_{\gamma k}^\mu + c_{\gamma\mu}^k) \tag{4.5}$$
$$\Gamma_{i\tau}^{\ \beta} = {}^1/_2 a^{\beta j}(\partial a_{ij}/\partial x_\tau + \partial a_{\tau j}/\partial x_j - \partial a_{i\tau}/\partial x_j)$$

The commutative case $c_{jk}^{\ i} = 0$. We will find the compatibility conditions for system (4.5). For each pair of equations

$$\partial\alpha_k^{\ i}/\partial x_\tau = \Gamma_{i\tau}^{\ \beta}\alpha_k^{\ \beta}, \quad \partial\alpha_k^{\ i}/\partial x_\sigma = \Gamma_{i\sigma}^{\ \beta}\alpha_k^{\ \beta}$$

we obtain

$$R_{\sigma i\tau}^\gamma \equiv \partial\Gamma_{i\tau}^{\ \gamma}/\partial x_\sigma - \partial\Gamma_{i\sigma}^{\ \gamma}/\partial x_\tau + \Gamma_{\beta\sigma}^\gamma\Gamma_{i\tau}^{\ \beta} - \Gamma_{\beta\tau}^\gamma\Gamma_{i\sigma}^{\ \beta} = 0$$

Using the Christoffel symbol of the first kind $\Gamma_{j,i\tau}$, we can express these conditions in terms of the Riemann curvature tensor with dropped index

$$R_{mi\tau\sigma} \equiv \frac{\partial\Gamma_{m,i\tau}}{\partial x_\sigma} - \frac{\partial\Gamma_{m,i\sigma}}{\partial x_\tau} + a^{j\beta}(\Gamma_{j,i\sigma}\Gamma_{\beta,m\tau} - \Gamma_{j,i\tau}\Gamma_{\beta} \quad) = 0 \tag{4.6}$$
$$\Gamma_{j,i\tau} = \frac{1}{2}\left(\frac{\partial a_{ij}}{\partial x_\tau} + \frac{\partial a_{\tau j}}{\partial x_i} - \frac{\partial a_{i\tau}}{\partial x_j}\right), \quad \Gamma_{i\tau}^{\ \beta} = a^{j\beta}\Gamma_{j,i\tau}$$

This is the well-known condition for isometry of the Riemann space with metric $ds^2 = a_{ij}dx_i dx_j$ to Euclidean space. The number of algebraically independent components of the Riemann tensor is $s^2(s^2-1)/12$. They have the symmetry properties

$$R_{mi\tau\sigma} = -R_{im\tau\sigma} = -R_{mi\sigma\tau} = R_{\tau\sigma mi}, \; R_{mi\tau\sigma} + R_{m\tau\sigma i} + R_{m\sigma i\tau} = 0.$$

It can be shown that (4.4) are particular integrals of Eqs.(4.5). This means that, if conditions (4.6) hold, then the functions $\alpha_k^{\ i}$ can be evaluated by finding the general solution of system (4.5), while the constants of this solution are found from conditions (4.4).

Example. Consider the inertial motion of an autonomous system with two degrees of freedom. With $s = 2$, conditions (4.6) reduce to the single condition

$$R_{1212} = 0 \tag{4.7}$$

The motion of the system can be expressed as the motion of a material particle over a surface with the metric

$$ds^2 = a_{11}dx_1^2 + 2a_{12}dx_1dx_2 + a_{22}dx_2^2$$

If condition (4.7) holds, the metric becomes Euclidean. The corresponding coordinate transformation is evaluated by quadratures. We also solve in quadratures the corresponding mechanical problem, i.e., evaluate the trajectories (geodesics), the two linear integrals and the general solution.

For instance, let the surface over which the particle moves be a smooth cone. The kinetic energy of the particle in spherical coordinates $\varphi \equiv x_1,\ \alpha \equiv \alpha(x_1),\ R \equiv x_2$ is

$$T = {}^1/_2\,[x_1^2\,(\alpha'^2\,(x_1) + \sin^2\alpha\,(x_1))\,x_1^{\cdot 2} + x_2^{\cdot 2}] \qquad (m = 1)$$

Condition (4.7) is satisfies for any function $\alpha(x_1)$ (it is assumed to be 2π-periodic, $0 < \alpha_0 \leqslant \alpha < \pi/2$), since the cone is a developable surface. The change of variables

$$x_1' = \int x_2\,(\alpha'^2 + \sin^2\alpha)\cos\psi dx_1 + \sin\psi dx_2, \quad \psi = \int (\alpha'^2 + \sin^2\alpha)^{1/2}\,dx_1$$
$$x_2' = \int x_2\,(\alpha'^2 + \sin^2\alpha)\sin\psi dx_1 - \cos\psi dx_2$$

reduces the kinetic energy to the canonical form $T = {}^1/_2\,(x_1'^{\cdot 2} + x_2'^{\cdot 2})$.

All the quantities of the dynamic problem can also be evaluated by quadratures. In order, to deal solely with explicit and elementary functions, we confine ourselves to the case of a right circular cone $(\alpha = \text{const})$. The equations of the trajectories are in this case

$$R = a\,(b\sin\psi + c\cos\psi)^{-1}, \quad \psi = \varphi\sin\alpha$$

The linear integrals and complete integral of the Hamilton-Jacobi equation are

$$R\varphi^{\cdot}\cos\psi + R^{\cdot}\sin\psi/\sin\alpha = c_1, \quad R\varphi^{\cdot}\sin\psi - R^{\cdot}\cos\psi/\sin\alpha = c_2 \tag{4.8}$$
$$V = \sqrt{2h}R\sin(\psi - \varphi_0);\ h,\ \varphi_0 = \text{const}$$

The non-commutative case. The compatibility conditions for system (4.5) can be written at once:

$$R_{\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4} = C_{\beta_1\beta_2\beta_3\beta_4}\alpha_{\beta_1}{}^{\varepsilon_1}\alpha_{\beta_2}{}^{\varepsilon_2}\alpha_{\beta_3}{}^{\varepsilon_3}\alpha_{\beta_4}{}^{\varepsilon_4} \tag{4.9}$$
$$C_{\beta_1\beta_2\beta_3\beta_4} = \overset{*}{c}{}^{\gamma}_{\beta_3\beta_2}\overset{*}{c}{}^{\beta_1}_{\beta_4\gamma} + c^{\gamma}_{\beta_3\beta_4}\overset{*}{c}{}^{\beta_1}_{\gamma\beta_2} - \overset{*}{c}{}^{\gamma}_{\beta_4\beta_2}\overset{*}{c}{}^{\beta_1}_{\beta_3\gamma} \tag{4.10}$$

(for correct tensor arrangement of the indices the super- and sub-scripts of the functions α_j^i must change places).

The constant tensor $C_{\beta_1\beta_2\beta_3\beta_4}$ has the same symmetry properties as the Riemann tensor

$$C_{\beta_1\beta_2\beta_3\beta_4} = -C_{\beta_2\beta_1\beta_3\beta_4} = -C_{\beta_1\beta_2\beta_4\beta_3} = C_{\beta_3\beta_4\beta_1\beta_2},$$
$$\sum_{(\beta_2,\beta_3,\beta_4)} C_{\beta_1\beta_2\beta_3\beta_4} = 0,$$

which follow from (4.10).

As in the commutative case, relations (4.4) are integral manifolds of Eqs.(4.5). With $s = 2$, calculations give

$$R_{1212} = C_{\beta_1\beta_2\beta_3\beta_4}\alpha_{\beta_1}{}^1\alpha_{\beta_2}{}^2\alpha_{\beta_3}{}^1\alpha_{\beta_4}{}^2 = -\,[(c_{12}{}^1)^2 + (c_{12}{}^2)^2]\,(\alpha_1{}^1\alpha_2{}^2 - \alpha_2{}^1\alpha_1{}^2)^2$$

Since $(\alpha_1{}^1\alpha_2{}^2 - \alpha_2{}^1\alpha_1{}^2)^2 = a_{11}a_{22} - a_{12}{}^2$, conditions (4.9) reduce to the single condition

$$R_{1212} = -\,[(c_{12}{}^1)^2 + (c_{12}{}^2)^2]\,(a_{11}a_{22} - a_{12}{}^2) \tag{4.11}$$

which describes a surface of constant negative curvature. An example is the pseudosphere (formed by rotation of a tractrix).

In short, the equations of motion of a mechanical system with two degrees of freedom, the coefficients of whose kinetic energy satisfy condition (4.11), by the Theorem of Para.3, admit of two linear integrals.

In problems with $s \geqslant 3$ coditions (4.9) introduce new (apart from relations (4.4)) finite constraints on the functions α_j^i. This prevents us from obtaining useful expressions for the compatibility conditions of system (4.5) in terms of a single tensor a_{ij} and its derivatives (i.e., relations which do not contain unknown functions α_j^i). On the other hand, we are sometimes able to find solutions of Eqs.(4.5) without integrating them.

Let us describe one such method. Using (4.9), we evaluate the components of the Ricci tensor

$$R_{\varepsilon_2\varepsilon_4} = a^{\varepsilon_1\varepsilon_3}R_{\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4} = C_{\beta_1\beta_2\beta_3\beta_4}\alpha^{\varepsilon_1}_{\beta_1}\alpha^{\varepsilon_2}_{\beta_2}\alpha^{\varepsilon_3}_{\beta_3}\alpha^{\varepsilon_4}_{\beta_4}\xi^{\gamma}_{\varepsilon_1}\xi^{\gamma}_{\varepsilon_3} = h_{\beta_2\beta_4}\alpha^{\varepsilon_2}_{\beta_2}\alpha^{\varepsilon_4}_{\beta_4}$$
$$h_{\beta_2\beta_4} = \sum_{\beta=1}^{s} C_{\beta\beta_2\beta\beta_4}$$

Hence, using the relations $\xi_k{}^l = a^{ik}\alpha_l{}^i$, $\xi_j{}^i\,\alpha_k{}^j = \delta_k{}^i$, we obtain $a^{i\gamma}R_{k\gamma}\alpha_l{}^i = h_{\gamma l}\alpha_\gamma{}^k$, in the matrix form

$$R^*\alpha = \alpha h; \quad R^* = (a^{\gamma\beta}R_{k\beta}), \quad \alpha = (\alpha_l^{\gamma}), \quad h = (h_{\gamma l}) \tag{4.12}$$

It follows from the similarity of matrices R^* and h that the eigenvalues of R^*, and hence the coefficients of its characteristic equation, must be constant. (In practice, these necessary conditions have to be checked in the first place). In view of this, not more than $s(s+1)/2 - s = s(s-1)/2$ of Eqs. (4.12) (in which R^* and h are symmetric matrices) will be algebraically independent. In the general position the number of independent equations cannot be less than $s(s-1)/2$. Together with the $s(s+1)/2$ Eqs. (4.4) we obtain s^2 equations for evaluating the functions α_j^i. Substitution of the functions thus obtained into Eqs. (4.5) gives the necessary and sufficient conditions for our problem to be solvable.

5. Motion under the action of active forces. We now turn to the case when the mechanical system is subject to the action of both conservative and non-conservative forces. The latter will be assumed for simplicity to be independent of the velocities, i.e., they are positional displacements. As before, the constraints are assumed to be holonomic and non-stationary, and the kinetic energy, transformed to Chetayev variables, is assumed to be independent of the coordinates (see (1.9)). However, when forces are present, we need not assume that the function b_0 appearing in the expression for H^* in accordance with the second of (4.1) is independent of the coordinates; it can be associated with the potential energy $U(x)$, and the forces corresponding to the sum $b+U$ to belong to non-conservative forces. Then, the forces obtained will be called reduced, and denoted by Q_α'. Obviously, $Q_\alpha' = Q_\alpha - \partial(b_0+U)/\partial x_\alpha$. We assume that, after introducing a suitable algebra and forces Q_α'

$$H^* = {}^1/_2 b_{ij}(t)\, y_i y_j + b_i(t)\, y_i \tag{5.1}$$

The commutation relations for this case are (1.6) and (1.7).

Relations (1.7) show that the necessary and sufficient conditions for the system of operators X_0^*, X_k^* to remain closed when reduced forces are present are

$$X_k Q_i' = -Q_\alpha' \partial \xi_\alpha^{\ k}/\partial x_i$$

which can be given the equivalent form

$$X_i Q_k^* = c_{ik}^{\ \ l} Q_l^*, \quad Q_k^* = \xi_\alpha^{\ k} Q_\alpha'$$

Since it can be shown in the usual way that this system is compatible, it has solutions, which can be evaluated in quadratures if the algebra A is solvable (see Para.3).

Let us find the conditions under which the forces Q_α' admit of the force function (possibly time-dependent)

$$Q_\alpha' = \partial U'/\partial x_\alpha, \quad Q_k^* = \xi_\alpha^{\ k} \partial U'/\partial x_\alpha = X_k U'$$

By conditions (5.1),

$$X_j X_k U' - c_{jk}^l X_l U' = (X_j X_k - c_{jk}^l X_l) U' = X_k X_j U' = X_k Q_j^* = 0 \tag{5.2}$$

$$Q_j^* = c_j^*(t), \quad c_j^* c_{ik}^j = 0$$

Conditions (5.2) are in fact sufficient for the forces Q_α' to admit of a force function. Indeed, $Q_k' = c_l^* \alpha_l^{\ k}$. On multiplying Maurer's equations by c_l^* and summing over l, we obtain

$$\frac{\partial c_l^* \alpha_l^{\ \mu}}{\partial x_\nu} - \frac{\partial c_l^* \alpha_l^{\ \nu}}{\partial x_\mu} = \frac{\partial Q_\mu'}{\partial x_\nu} - \frac{\partial Q_\nu'}{\partial x_\mu} = c_l^* c_{jk}^l \alpha_j^{\ \mu} \alpha_k^{\ \nu} = 0$$

Hence $Q_\alpha' = \partial U'/\partial x_\alpha$, $U' = U'(t, x)$.

We define the functions $\mu_k^{\ i}(t, x)$ by (2.2): $X_0 \mu_k^{\ i} = c_{p0}^{\ \ k} \mu_p^{\ i}$, $X_\nu \mu_k^{\ i} = c_{p\nu}^{\ \ k} \mu_p^{\ i}$. We examine the properties of the products $\varepsilon^i = \mu_k^{\ i} Q_k^*$. We have

$$X_\nu \varepsilon^i = \mu_k^{\ i} X_\nu Q_k^* + Q_k^* X_\nu \mu_k^{\ i} = \mu_k^{\ i} c_{\nu k}^l Q_l^* + Q_k^* c_{p\nu}^k \mu_p^{\ i} = (c_{\nu k}^l + c_{k\nu}^l) \mu_k^{\ i} Q_l^* = 0$$

Hence $\partial \varepsilon^i/\partial x_j = 0$, $\mu_k^{\ i} Q_k^* = \varepsilon^i(t)$. Thus the most general form of the Q_α', corresponding to admissible forces, is given, apart from the arbitrariness of the functions $\varepsilon(t)$, by the constraints imposed on the mechanical system

$$Q_l^* = \varepsilon^i(t)\, \mu_i^{*l}, \quad \mu_i^{*l} \mu_k^{\ i} = \delta_k^{\ l}$$

We shall seek the linear integrals of the mechanical system in the form $\omega_i' = \mu_j^{\ i} y_j + \mu_0^{\ i}(t)$. On substituting these expressions into the equations $X_1^* \omega_1' = 0, \ldots, X_s^* \omega_i' = 0$, we get

$$X_k^* \omega_i' = y_j X_k \mu_j^{\ i} + \mu_j^{\ i} c_{kj}^p y_p = (c_{pk}^j + c_{kp}^j) \mu_p^{\ i} y_j = 0$$

By choosing the functions $\mu_0^{\ i}(t)$ we try to satisfy the equations $X_0^* \omega_i' = 0$

$$X_0{}^*\omega_i{}' = y_i X_0 \mu_j{}^i + \mu_j{}^i (c_{0j}{}^p y_p + Q_j{}^*) + d\mu_0{}^i (t)/dt =$$
$$(c_{p0}^j + c_{0p}^j)\, \mu_p{}^i y_j + \mu_j{}^i Q_j{}^* + d\mu_0{}^i (t)/dt = \varepsilon^i (t) + d\mu_0{}^i (t)/dt$$

Obviously, the equations will be satisfied if we put

$$\mu_0{}^i (t) = -\int \varepsilon^i (t)\, dt$$

Thus the required linear integrals are

$$\omega_i{}' = \mu_j{}^i (t, x)\, y_j - \int \varepsilon^i (t)\, dt = c_i$$

Since $\mu_j{}^i y_j = \omega_i$ are the first integrals of the mechanical system when there are no admissible forces, the imposition of the latter leads to evolution of these integrals $\omega_i = \int \varepsilon^i (t)\, dt + c_i$.

Notice that, if the basic algebra is commutative, conditions (5.2) are satisfied, and the motion takes place in a field with a force function. In this case $\mu_j{}^i = \mu_j{}^i (t)$. If $\mu_j{}^i = \delta_j{}^i$, then $\varepsilon^j = c_j{}^*$. The first integrals are

$$\omega_i{}' = y_i - \int c_i{}^*\, dt = c_i \tag{5.3}$$

and the force function is given by a quadrature

$$V \equiv U' = c_j{}^* \int \alpha_j{}^l\, dx_l \tag{5.4}$$

This may not be a unique function of the system coordinates. The problem of the motion under the action of admissible forces then losses its mechanical meaning.

Example. In the problem of particle motion over the surface of a right circular cone

$$\alpha_1{}^1 = x_2 \cos\psi,\ \alpha_1{}^2 = \sin\psi/\sin\alpha,\ \alpha_2{}^1 = x_2 \sin\psi$$
$$\alpha_2{}^2 = -\cos\psi/\sin\alpha,\ x_1 = \varphi,\ x_2 = R\sin\alpha,\ \psi = x_1 \sin\alpha$$

Here we have the commutative case. Calculation from (5.4) gives

$$V = R\,[c_1{}^* (t) \sin(\varphi \sin\alpha) - c_2{}^* (t) \cos(\varphi \sin\alpha)]$$

The force function V is unique only when $\alpha = \pi/2$, when the particle moves over a plane in a homogeneous field of force. When $\alpha \neq \pi/2$ the forces acting on the particle lose their mechanical nature, but the equations of motion, in accordance with (5.3), admit of linear integrals corresponding to (4.8) with c_1, c_2 replaced by $c_1 - \int c_1{}^* dt,\ c_2 - \int c_2{}^* dt$.

6. Remark on the nature of the equations of motion.

Systems of hydrodynamic type. If the Hamiltonian of the system can be reduced to the form (5.1), the equations of motion can be written in the form ($Q_l{}'$ are the reduced forces).

$$x_\alpha{}^\cdot = b_{ij} \xi_\alpha{}^i y_j + \xi_\alpha{}^i b_i + \xi_\alpha \tag{6.1}$$
$$y_j{}^\cdot = c_{lj}^p b_{li} y_p y_i + (c_{lj}^p b_l + c_{0j}^p)\, y_p + \xi_l{}^j Q_l{}' \tag{6.2}$$

A typical feature of this description of the mechanical problem is that, with $Q_j{}^* = c_j{}^* (t)$, the equations for the Chetayev variables y_j are separate from the rest, and can therefore have an independent origin. For instance, with $b_{li} = \delta_i{}^l,\ b_l = c_{0j}{}^p = Q_l{}' = 0$, Eqs.(6.2) can be written as

$$y_j{}^\cdot = A_{jlk} y_l y_k, \qquad A_{ilk} = {}^1/_2 (c_{ki}^l + c_{li}^k)$$

It can be verified that $A_{ilk} = A_{ikl},\ A_{ilk} + A_{lki} + A_{kil} = 0$. If the extra conditions $c_{il}{}^l = 0\ (A_{ill} = 0)$, are imposed on the structural constants, we arrive at a system of hydrodynamic type /7/.

In cases when Eqs.(6.2) describe a separate problem, Eqs.(6.1) become auxiliary. The components of the operators X_0, X_k in them can be chosen arbitrarily provided that the structural constants are chosen in accordance with the specific problem. If the $Q_j{}^* = \xi_l{}^j Q_l{}'$ are then admissible quantities, and Eqs.(6.1) (noting that there are s linear integrals in the present case) are integrable, the problem can be completely solved.

In other cases, conversely, Eqs.(6.2) may be found to be integrable when admissible forces are present. Then, if the linear integrals can be solved with respect to the coordinates x_α, we obtain a complete solution of the mechanical problem described by system (6.1), (6.2).

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A CONSEQUENCE OF THE INVARIANCE OF THE GAUSS PRINCIPLE*

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The invariant form of the Gauss principle of least compulsion in the space of positions of a system with constraints (some of which may be non-holonomic) is considered. A modified construction of the compulsion function in configuration space is proposed. The modified expression contains information on the constraints. From the complete system of differential equations, equations are obtained for finding the reactions of the constraints. An example of the use of this approach is given.

Manay authors, see /1, 2/, have considered the analytic form of the Gauss principle. However, there is still no standard treatment of the principle in analytical dynamics. For instance, it is said in /3/, p.192, that "the Gauss principle ... does not have the analytical advantages of other principles", and "is of less value than the principle at least action (/3/, p.134). Other authors (/4/, p.219) say that the "Gibbs-Appell equations (with which the Gauss principle is closely linked) represent the simplest and at the same time the most general form of the equations of motion". Yet, though these equations are closely linked with the principle of least compulsion, they do not contain the compulsion function

$$Z=\frac{1}{2}\sum_{\nu} m_\nu\left(x^{\cdot\cdot}-\frac{X_\nu}{m_\nu}\right)^2 \tag{0.1}$$

but the Gibbs-Appell function

$$S=\frac{1}{2}\sum_{\nu} m_\nu x^{\cdot\cdot 2} \tag{0.2}$$

where m_ν is the mass of the ν-th point of the system, $x_\nu^{\cdot\cdot}$ are the coordinates of the acceleration vector, and X_ν are the coordinate of the vector of forces in a rectangular orthogonal system of coordinates.

Apart from these inconsistencies, there are difficulties in introducing the generalized Lagrange coordinates, in which

$$Z = S - Q_\alpha q^{\cdot\cdot\alpha} \tag{0.3}$$

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